Maximum Number of Pairwise G-different Permutations

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Colliding permutations

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Example:

$$\begin{matrix}[1, & 2, & \mathbf{3}, & 4, & 5] \\ [5, & 4, & \mathbf{2}, & 1, & 3]\end{matrix} \qquad \begin{matrix}[1, & 2, & 3, & 4, & 5] \\ [1, & 2, & 5, & 4, & 3]\end{matrix}$$

Families of pairwise colliding permutations

- What is the maximum size of a family of pairwise colliding permutations of $1 \dots n$?

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- Example:

$n = 3$			$n = 4$			
			1	2	3	4
			1	2	4	3
2	1	3	3	1	2	4
1	2	3	4	1	2	3
1	3	2	2	3	1	4
			2	4	1	3

Known results

Theorem

The maximum size of a family of pairwise colliding permutations is at most $\binom{n}{\lfloor n/2 \rfloor}$.

- This upper bound is conjectured to be tight, and has been verified for $n < 8$.
- The best lower bound currently known is $\sim 1.7^n$.
  (Note that upper bound is $\sim 2^n$.)

G-different permutations

Definition

For any graph G, let two permutations σ and π of subsets of $V(G)$ be *G-different* if there exists a position i for which $(\sigma(i), \pi(i)) \in E(G)$.

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For any graph G with n vertices, let $F(G)$ be the maximum size of a family of pairwise G-different permutations.

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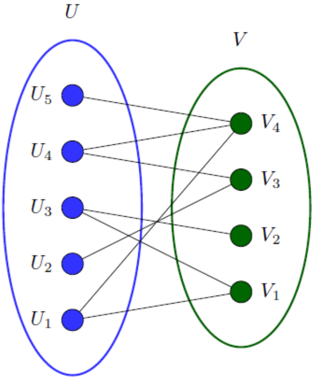
For any graph G with n vertices, let $F(G)$ be the maximum size of a family of pairwise G-different permutations.

- If $L(n)$ is a path on n vertices, then $L(n)$-different permutations are equivalent to colliding permutations.
- We want to bound $F(L(n))$ as $n \to \infty$.

Bipartite Graphs

Bipartite graph:



Complete bipartite graph: Each vertex in one set connected to every vertex in the other set.

Complete Bipartite Graphs: A Basic Result

Lemma

Let $K_{a,n-a}$ be a complete bipartite graph with a vertices on one side and $n-a$ vertices on the other. Then $F(K_{a,n-a}) = \binom{n}{a}$.

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Note that any bipartite graph G with a vertices on one side and $n-a$ on the other satisfies $F(G) \leq F(K_{a,n-a}) = \binom{n}{a}$

Focus of Our Project: Bipartite Graphs

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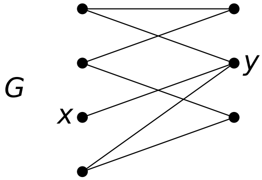
- Arbitrary graphs are too general: we have focused on bipartite graphs
- Path and complete bipartite graph are extremes:
  - Path has the least number of edges among connected bipartite graphs
  - Complete bipartite graph has the most
- Conjectured to have same maximal pairwise colliding family size
- What happens in the in-between cases?

Recursion for putting lower bounds on $F(G)$

Lemma

Let (x, y) be an edge in G. Then $F(G) \geq F(G - \{x\}) + F(G - \{y\})$.



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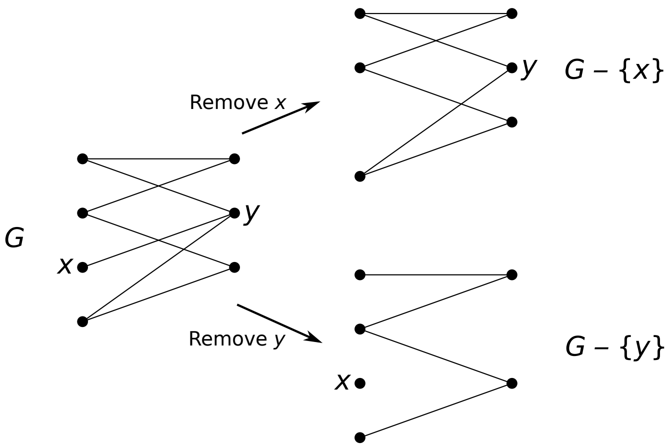

Recursion for putting lower bounds on $F(G)$

- Consider a family $\{\pi_1, \pi_2, \ldots, \pi_{F(G-\{x\})}\}$ of pairwise G-different permutations of $V(G-\{x\})$ and a family $\{\sigma_1, \sigma_2, \ldots, \sigma_{F(G-\{y\})}\}$ of pairwise G-different permutations of $V(G-\{y\})$.
- Then concatenate x and y back on:

$$\begin{array}{c} F(G-\{x\}) + F(G-\{y\}) \\ G\text{-different permutations} \\ \text{of } V(G) \end{array} \left\{ \begin{array}{l} \left.\begin{array}{ll} x & \pi_1 \\ x & \pi_2 \\ & \vdots \\ x & \pi_{F(G-\{x\})} \end{array}\right\} \begin{array}{c} F(G-\{x\}) \text{ permutations} \\ \text{of } V(G-\{x\}) \end{array} \\ \\ \left.\begin{array}{ll} y & \sigma_1 \\ y & \sigma_2 \\ & \vdots \\ y & \sigma_{F(G-\{y\})} \end{array}\right\} \begin{array}{c} F(G-\{y\}) \text{ permutations} \\ \text{of } V(G-\{y\}) \end{array} \end{array} \right.$$

Complete bipartite graph with matching removed

Theorem

Let $G(n, a)$ be the bipartite graph with n vertices, a of which are in the first subset and $n - a$ of which are in the second subset, such that $G(n, a)$ is complete with a maximal matching removed. Then for all $n \geq 3$,

$$F(G(n, a)) = \binom{n}{a}.$$

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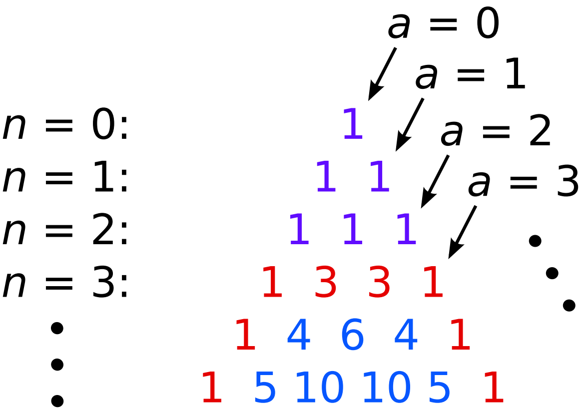
$$F(G(n, a)) = \binom{n}{a}.$$

- Strict equality in the above equation requires a sufficient base case.
- For the base case here, a simple construction gives $F(G(n, 1)) = F(G(n, 2)) = 3$.

Two variable recursion for bounding $F(G)$

- Recursion: $F(G(n, a)) \geq F(G(n-1, a-1)) + F(G(n-1, a))$.



Maximum degree of complement a constant

Definition

Let $F(n, a, \Delta_c)$ be the minimum value of $F(G)$ over all bipartite graphs G with n vertices, a of which are in the first subset, such that the maximum degree of the complement of G is Δ_c.

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Theorem

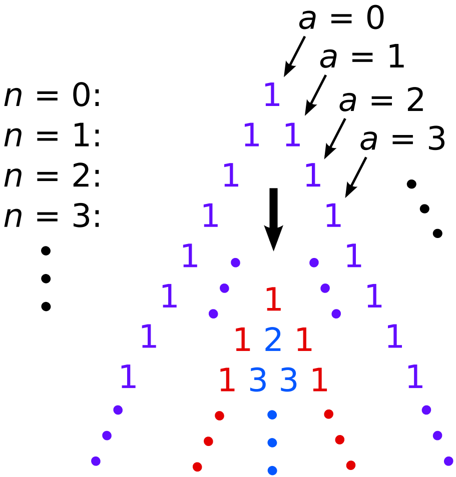
For any constant nonnegative integer Δ_c, there exists a constant s such that

$$F(n, a, \Delta_c) \geq s\binom{n}{a}$$

for all n and a.

Two variable recursion for bounding $F(G)$

- Removing a vertex cannot increase Δ_c.
- Recursion: $F(G(n,a)) \geq F(G(n-1,a-1)) + F(G(n-1,a))$.



Increasing the maximum degree of complement

Theorem

$$\lim_{n\to\infty} \frac{1}{n} \log_2 F\left(n, \left\lfloor \frac{n}{2} \right\rfloor, o(n)\right) = 1.$$

- Loosely speaking, this means that if $\Delta_c = o(n)$, then $F(G)$ grows on the order of 2^n as $n \to \infty$ when G is balanced ($a = n - a$).

Increasing the maximum degree of complement

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- Loosely speaking, this means that if $\Delta_c = o(n)$, then $F(G)$ grows on the order of 2^n as $n \to \infty$ when G is balanced $(a = n - a)$.
- This theorem makes us ask: how much can we increase Δ_c while keeping $F(G)$ on the order of 2^n?

Conjecture

$$\lim_{n\to\infty} \frac{1}{n} \log_2 F\left(n, \left\lfloor \frac{n}{2} \right\rfloor, \frac{n}{c}\right) = 1$$

where $c > 2$.

Union of disjoint graphs

- Previously known: If G_1 and G_2 are disjoint graphs, then

$$F(G_1 \cup G_2) \geq F(G_1) \cdot F(G_2).$$

- Therefore if $F(G_1) \approx 2^{|V(G_1)|}$ and $F(G_2) \approx 2^{|V(G_2)|}$ then

$$F(G_1 \cup G_2) \approx 2^{|V(G_1)|+|V(G_2)|} = 2^{|V(G_1 \cup G_2)|}.$$

- When formalized, this idea can be used to show that the union of many disjoint subgraphs is on the order of 2^n.

Formalizing the union of disjoint subgraphs

Theorem

Let G be a balanced bipartite graph consisting of the union of k disjoint balanced complete bipartite graphs $G_1, G_2, \ldots, G_k$. If

$$k = O\left(\frac{n}{\log_2 n}\right),$$

then

$$\lim_{n\to\infty} \frac{1}{n}\log_2 F(G) = 1.$$

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then

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Corollary

There exists a graph G with maximum degree $O(\log_2 n)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log_2 F(G) = 1.$$

Future Work

Conjecture

$$\lim_{n\to\infty} \frac{1}{n} \log_2 F\left(n, \left\lfloor \frac{n}{2} \right\rfloor, \frac{n}{c}\right) = 1$$

where $c > 2$.

Current strategy:

- Partition graph into large, balanced bi-cliques
- Apply union of disjoint subgraphs strategy
- Issue: removing large cliques disrupts structure of graph

In sparser graphs, large bi-cliques cannot always be found: must be partitioned into other types of subgraphs.

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